The concrete theory of numbers : New Mersenne conjectures. Simplicity and other wonderful properties of numbers $L(n) = 2^{2n} \pm 2^n \pm 1$.

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Abstract

New Mersenne conjectures. The problems of simplicity, common prime divisors and free from squares of numbers $L(n) = 2^{2n} \pm 2^n \pm 1$ are investigated. Wonderful formulas gcd for numbers $L(n)$ and numbers repunit are proved.

1 Introduction

In present work we consider sequences of integers of the following kind :

$$L_1(n) = 2^{2n} + 2^n + 1, \qquad (L_1)$$
$$L_2(n) = 2^{2n} + 2^n - 1, \qquad (L_2)$$
$$L_3(n) = 2^{2n} - 2^n + 1, \qquad (L_3)$$
$$L_4(n) = 2^{2n} - 2^n - 1, \qquad (L_4)$$

where $n \geq 1$ is integer.

For the numerical sequence, being the union of numerical sequences $L_1(n)$, $L_2(n)$, $L_3(n)$, $L_4(n)$, we use a designation

$$L(n) = 2^{2n} \pm 2^n \pm 1, \qquad (1)$$

where $n \geq 1$ is integer.

The author is interested to research the new Mersenne conjectures, concerning numbers $L(n)$. The reviews concerning Mersenne numbers and new Mersenne conjectures are available here [4, 6, 7, 9, 13].

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Will we use the following designations further :

$(a,b) = gcd(a,b)$ is the greatest common divider of integers $a > 0$, $b > 0$.

p, q are odd prime numbers.

$m \perp n \Longleftrightarrow m, n-$ are integers and $gcd(m,n) = 1$(see[4]).

If it is not stipulated specially, the integer positive numbers are considered.

We are interested by the following questions concerning numbers $L(n)$: question of simplicity of numbers, question of the common divisors and question of freedom from squares.

For numbers $L(n)$ two general simple statements are fair. These statements represent trivial consequences of the small Fermat's theorem , submitted by the following comparison,

$$n^{p^k} \equiv n^{p^{k-1}} (mod\, p^k) \text{ for } k > 0 \text{(see[4])}$$

or chain of comparisons

$$n^{p^{N+k}} \equiv n^{p^{N-1+k}} \equiv \ldots \equiv n^{p^k} \equiv n^{p^{k-1}} (mod\, p^k) \text{ for } k > 0,\ N \geq 0,$$

n of integers.

Statement 1. *If $L(l) \equiv 0 (mod\, p)$, where $l > 0$ is integer, p is prime number, then*

$$L((p-1) \cdot k + l) \equiv 0 (mod\, p),$$

where $k \geq 0$ is any integer.

Statement 2. *If $L(l) \equiv 0 (mod\, p^t)$, where $t > 0$, $l > 0$ is integer, p is prime number, then*

$$L(p^{N+t} - p^{t-1} + l) \equiv 0 (mod\, p^t),$$

where $N \geq 0$ is any integer.

2 Numbers $L_1(n) = 2^{2n} + 2^n + 1$

§1. Let's bring the simple statements concerning numbers $L_1(n)$.

Lemma 1. *For numbers $L_1(n) = 2^{2n} + 2^n + 1$ following statements are fair :*

(1) The first prime numbers $L_1(n)$ correspond to $n = 1, 3, 9$.
$L_1(1) = 7$, $L_1(3) = 73$, $L_1(9) = 262657$.

(2) If $n \geq 2$ is an even number, then number

$$L_1(n) \equiv 0 (mod\, 3) \tag{2}$$

is composite. If $n \geq 1$ is an odd number, then

$$L_1(n) \equiv 1 \not\equiv 0 (mod\, 3) \tag{3}$$

(3) If $n > 1$, $n \not\equiv 0(mod\, 3)$, then number

$$L_1(n) \equiv 0(mod\, 7) \tag{4}$$

is composite.

Proof. Validity of congruences (2)-(4) obviously follows from trivial comparisons $2 \equiv -1(mod\, 3)$, $2^3 \equiv 1(mod\, 7)$. □

Thus, prime numbers $L_1(n)$ are probable only for $n = 3k$, where k is odd number.

Lemma 2. *Let $k > 1$ is an integer. Then there will be a prime number $q > 3$, such that number 2 on the module q belongs to a index 3^k.*

Proof. Let's consider expression
$A = 2^{3^k} - 1 = (2^{3^{k-1}})^3 - 1 = (2^{3^{k-1}} - 1) \cdot ((2^{3^{k-1}})^2 + 2^{3^{k-1}} + 1)$.
Let q is a prime number such that
$B = (2^{3^{k-1}})^2 + 2^{3^{k-1}} + 1 \equiv 0(mod\, q)$. If $q = 3$, then $B \equiv 1 \not\equiv 0(mod\, 3)$.
Hence $q > 3$, $2^{3^k} - 1 \equiv 0(mod\, q)$. Let's assume, that
$2^{3^l} - 1 \equiv 0(mod\, q)$, where $k > l \geq 0$. Let $d = k - 1 - l \geq 0$, $3^d \geq 1$.
Then $(2^{3^l})^{3^d} \equiv 1(mod\, q)$, $2^{3^{l+d}} \equiv 1(mod\, q)$, $2^{3^{k-1}} \equiv 1(mod\, q)$,
$B = (2^{3^{k-1}})^2 + 2^{3^{k-1}} + 1 \equiv 3 \equiv 0(mod\, q)$, $q = 3$. Have received the contradiction. It is obvious, that $d = 3^k$ is the least positive number, for which the comparison $2^d - 1 \equiv 0(mod\, q)$ is feasible. □

§2. In connection with the lemma 1 the research of prime divisors of numbers $L_1(n)$ is interesting, where $n = 3^k t$, where $k \geq 1$ is integer, $t \geq 1$ is an odd number. Since only at such n the numbers $L_1(n)$ can be *suspicious* on prime numbers. In the following theorem the interesting property gcd for numbers $L_1(n)$ is proved.

Theorem 1 (gcd of numbers L_1). *Let $k \geq 0$, $k_1 \geq 0$, $k_2 \geq 0$ be integers; $t_1 \geq 1$, $t_2 \geq 1$ are odd numbers, $t_1 \not\equiv 0(mod\, 3)$, $t_2 \not\equiv 0(mod\, 3)$. Then the following statements are fair :*

$$gcd(L_1(3^k t_1), L_1(3^k t_2)) = L_1(3^k gcd(t_1, t_2)). \tag{5}$$

$$gcd(L_1(3^{k_1} t_1), L_1(3^{k_2} t_2)) = 1 \tag{6}$$

at $k_1 \neq k_2$.

Proof. Let $t_3 = gcd(t_1, t_2)$, $t_1 = t_3 d_1$, $t_2 = t_3 d_2$, where $(d_1, d_2) = 1$, $d_1 \geq 1$, $d_2 \geq 1$ are odd numbers.

1) Let's prove equality (5). Let's consider the following formulas

$$\begin{aligned} A = L_1(3^k t_1) = 2^{2\cdot 3^k t_1} + 2^{3^k t_1} + 1, \quad & A(2^{3^k t_1} - 1) = 2^{3^{k+1} t_3 d_1} - 1, \\ B = L_1(3^k t_2) = 2^{2\cdot 3^k t_2} + 2^{3^k t_2} + 1, \quad & B(2^{3^k t_2} - 1) = 2^{3^{k+1} t_3 d_2} - 1, \\ C = L_1(3^k t_3) = 2^{2\cdot 3^k t_3} + 2^{3^k t_3} + 1, \quad & C(2^{3^k t_3} - 1) = 2^{3^{k+1} t_3} - 1. \end{aligned} \tag{7}$$

Then the following formulas are fair

$$A(2^{3^k t_1}-1)=C(2^{3^k t_3}-1)[2^{3^{k+1}t_3(d_1-1)}+2^{3^{k+1}t_3(d_1-2)}+\dots$$
$$\dots+2^{3^{k+1}t_3}+1], \quad (8)$$

$$B(2^{3^k t_2}-1)=C(2^{3^k t_3}-1)[2^{3^{k+1}t_3(d_2-1)}+2^{3^{k+1}t_3(d_2-2)}+\dots$$
$$\dots+2^{3^{k+1}t_3}+1]. \quad (9)$$

Let $q>1$ is prime number such, that

$$2^{3^k t_1}-1\equiv 0(mod\, q),\ C\equiv 0(mod\, q). \quad (10)$$

Then it follows from (7) and (10), that

$$2^{3^k t_3 d_1}-1\equiv 0(mod\, q),\ 2^{3^{k+1}t_3}-1\equiv 0(mod\, q).$$

Let's consider number $b=2^{3^k t_3}$. If $b\equiv 1(mod\, q)$, then it follows from (7) and (10), that $C\equiv 3\equiv 0(mod\, q)$, i.e. $q=3$. Since t_3 is odd number, then it follows from lemma 1, that $C\not\equiv 0(mod\, 3)$. We have come to the contradiction.

Thus, $b\not\equiv 1(mod\, q)$. Then the number b on the module q belongs to index $l_0>1$, $b^{l_0}\equiv 1(mod\, q)$. As $b^{d_1}\equiv b^3\equiv 1(mod\, q)$, that $d_1\equiv 3\equiv 0(mod\, l_0)$, $d_1\equiv 0(mod\, 3)$. Have received the contradiction.

We have proved, that $gcd(C,2^{3^k t_1}-1)=gcd(C,2^{3^k t_2}-1)=1$, hence, $gcd(A,B)\equiv 0(mod\, C)$. It is necessary to prove the opposite: if $d\mid gcd(A,B)$, then $d\mid C$, where $d>1$ an integer.

Let's assume, that there is an integer $d>1$ such, that

$$A\equiv B\equiv 0(mod\, d),\ but\ gcd(C,d)=1. \quad (11)$$

Then it follows from (7) and (11), that

$$2^{3^{k+1}t_3 d_1}-1\equiv 0(mod\, d),\ 2^{3^{k+1}t_3 d_2}-1\equiv 0(mod\, d).$$

Let $l_0>1$ is an index, to which the number $\mathbf{2}$ belongs on the module d, $2^{l_0}-1\equiv 0(mod\, d)$. Then $(3^{k+1}t_3)d_1\equiv(3^{k+1}t_3)d_2\equiv 0(mod\, l_0)$, $3^{k+1}t_3\equiv 0(mod\, l_0)$. Then from (7) $C(2^{3^k t_3}-1)\equiv 0(mod\, d)$, $2^{3^k t_3}-1\equiv 0(mod\, d)$, $A\equiv 3\equiv 0(mod\, d)$, $d=3$. Have received the contradiction. The equality (5) is proved.

2) Let's prove equality (6). Let's consider the following formulas

$$A_1=2^{2\cdot 3^{k_1}t_1}+2^{3^{k_1}t_1}+1,\quad A_1(2^{3^{k_1}t_1}-1)=2^{3^{k_1+1}t_1}-1,$$
$$A_2=2^{2\cdot 3^{k_2}t_2}+2^{3^{k_2}t_2}+1,\quad A_2(2^{3^{k_2}t_2}-1)=2^{3^{k_2+1}t_2}-1, \quad (12)$$

where $k_1\neq k_2$, $k_1<k_2$.

Let's assume, that $q > 1$ is a prime number such, that $A_1 \equiv A_2 \equiv 0(mod\, q)$, $2^{3^{k_1+1}t_1} \equiv 2^{3^{k_2+1}t_2} \equiv 1(mod\, q)$. Let $l_0 > 1$ is an index, to which the number 2 belongs on the module q, $2^{l_0} - 1 \equiv 0(mod\, q)$. Then $3^{k_1+1}t_1 \equiv 3^{k_2+1}t_2 \equiv 0(mod\, l_0)$. Since $k_1 + 1 \leq k_2$, then $3^{k_2}t_2 \equiv 0(mod\, l_0)$, $A_2 \equiv 3 \equiv 0(mod\, q)$, $q = 3$. Have received the contradiction. The theorem 1 is proved. □

Corollary 1. *If $n = 3^k t$, where $k \geq 1$ is integer, $t > 1$ is an odd number, $t \not\equiv 0(mod\, 3)$, then*

$$L_1(3^k t) \equiv 0(mod\, L_1(3^k))$$

is always composite number.

The summary of the received results concerning numbers L_1.

Theorem 2 (About numbers L_1)**.** *For numbers $L_1(n) = 2^{2n} + 2^n + 1$ the following statements are fair :*

(1) The prime numbers $L_1(1) = 7$, $L_1(3) = 73$, $L_1(9) = 262657$ are known.

(2) If $n \neq 3^k$, where $k \geq 0$ is an integer, then $L_1(n)$ is a composite number.

From the identity $2^{3^{k+1}} - 1 = (2^{3^k} - 1)[(2^{3^k})^2 + 2^{3^k} + 1]$ the following equality is received for numbers L_1

$$2^{3^{k+1}} - 1 = L_1(1) \cdot L_1(3) \cdot \ldots \cdot L_1(3^k), \tag{13}$$

where $k \geq 0$ is integer.

From the statement (6) of the theorem 1 the property follows

$$L_1(3^i) \perp L_1(3^j) \tag{14}$$

at $i \neq j$.

§3. The numbers L_1 are not free from squares, that is confirmed by the following examples $L_1(7) \equiv 0(mod\, 7^2)$, $L_1(104) \equiv 0(mod\, 13^2)$, $L_1(114) \equiv 0(mod\, 19^2)$. The following theorem takes place:

Theorem 3. *Let $k \geq 0$, $n \not\equiv 0(mod\, 3)$ are integers. Then the comparison is fair*

$$L_1(7^k n) \equiv 0(mod\, 7^{k+1}). \tag{15}$$

Proof. Let's consider number $B = 2^{7^k n} - 1$. Since $n \not\equiv 0(mod\, 3)$, then $B \not\equiv 0(mod\, 7)$.

Let $A = B \cdot L_1(7^k n) = 2^{7^k 3n} - 1$. Let's prove by induction on $k \geq 0$, that

$$2^{7^k 3n} - 1 \equiv 0(mod\, 7^{k+1}). \tag{16}$$

The case $k = 0$ is obvious. Lets make the inductive assumption, that for $k \leq m-1$ the comparison (16) is fair. Let's consider expression

$$2^{7^m 3n} - 1 = (2^{7^{m-1}3n})^7 - 1 =$$
$$= (2^{7^{m-1}3n} - 1) \cdot \sum_{i=0}^{6} (2^{7^{m-1}3n})^i \equiv 0 (mod\, 7^m \cdot 7) \equiv 0 (mod\, 7^{m+1}).$$

□

3 Numbers $L_2(n) = 2^{2n} + 2^n - 1$

First five prime numbers $L_2(1) = 5$, $L_2(2) = 19$, $L_2(3) = 71$, $L_2(4) = 271$, $L_2(6) = 4159$. From **the statement 1** validity of the comparisons follows

$$L_2(4k+1) \equiv 0 (mod\, 5),\; L_2(10k+7) \equiv L_2(10k+8) \equiv 0 (mod\, 11),$$

where $k \geq 0$ is integer.

The numbers $L_2(n)$ are not free from squares $L_2(68) \equiv 0 (mod\, 11^3)$, $L_2(97) \equiv 0 (mod\, 11^2)$.

The author has checked up the following worthy to attention facts for numbers $L_2(n)$.

1) For prime numbers $p \leq 5003$ the prime numbers $L_2(p) = 2^{2p} + 2^p - 1$ exist only for $p = 2$, $L_2(2) = 19$; $p = 3$, $L_2(3) = 71$; $p = 379$.

2) If we consider numbers $L_2(2^n) = 2^{2^{n+1}} + 2^{2^n} - 1$, then prime numbers $L_2(2^n)$ for $n \leq 17$ exist at $n = 1$, $L_2(2) = 19$; $n = 2$, $L_2(4) = 271$; $n = 4$, $L_2(16) = 4295032831$.

4 Numbers $L_3(n) = 2^{2n} - 2^n + 1$

Trivial property of numbers $L_3(n)$: if $n > 0$ is an even number, then

$$L_3(n) \equiv 1 \not\equiv 0 (mod\, 3), \tag{17}$$

if $n > 0$ is an odd number, then

$$L_3(n) \equiv 0 (mod\, 3). \tag{17'}$$

From **the statement 1** validity of comparisons follows

$$L_3(2(6k+1)) \equiv L_3(2(6k+5)) \equiv 0 (mod\, 13),$$

where $k \geq 0$ is integer.

The numbers $L_3(n)$ are not free from squares $L_3(2 \cdot 13) \equiv L_3(10 \cdot 13) \equiv 0 (mod\, 13^2)$, $L_3(3 \cdot 19) \equiv 0 (mod\, 19^2)$.

In the following theorem the interesting property of gcd for numbers L_3 is proved.

Theorem 4 (gcd of numbers L_3)**.** *Let $m \geq 0$, $m_1 \geq 0$, $m_2 \geq 0$, $n > 0$, $n_1 > 0$, $n_2 > 0$ are integers; $t_1 \geq 1$, $t_2 \geq 1$ are odd numbers, $t_1 \not\equiv 0(mod\, 3)$, $t_2 \not\equiv 0(mod\, 3)$. Then the statements are fair :*

$$gcd(L_3(3^m 2^n t_1), L_3(3^m 2^n t_2)) = L_3(3^m 2^n gcd(t_1, t_2)). \tag{18}$$

$$gcd(L_3(3^{m_1} 2^{n_1} t_1), L_3(3^{m_2} 2^{n_2} t_2)) = 1 \tag{19}$$

for $m_1 \neq m_2$ or $n_1 \neq n_2$.

Proof. Let $t_3 = gcd(t_1, t_2)$, $t_1 = t_3 d_1$, $t_2 = t_3 d_2$, where $(d_1, d_2) = 1$, $d_1 \geq 1$, $d_2 \geq 1$ are odd numbers.

1) Let's prove equality (18). Let's consider the following formulas

$$\begin{gathered} A = L_3(3^m 2^n t_1) = 2^{2\cdot 3^m 2^n t_1} - 2^{3^m 2^n t_1} + 1, \\ A(2^{3^m 2^n t_1} + 1) = 2^{3^{m+1} 2^n t_3 d_1} + 1, \\ B = L_3(3^m 2^n t_2) = 2^{2\cdot 3^m 2^n t_2} - 2^{3^m 2^n t_2} + 1, \\ B(2^{3^m 2^n t_2} + 1) = 2^{3^{m+1} 2^n t_3 d_2} + 1, \\ C = L_3(3^m 2^n t_3) = 2^{2\cdot 3^m 2^n t_3} - 2^{3^m 2^n t_3} + 1, \\ C(2^{3^m 2^n t_3} + 1) = 2^{3^{m+1} 2^n t_3} + 1. \end{gathered} \tag{20}$$

Then the formulas are fair

$$\begin{aligned} A(2^{3^m 2^n t_1} + 1) = C(2^{3^m 2^n t_3} + 1)[2^{3^{m+1} 2^n t_3 (d_1 - 1)} - \\ -2^{3^{m+1} 2^n t_3 (d_1 - 2)} + \dots \\ \dots + 2^{3^{m+1} 2^n t_3 2} - 2^{3^{m+1} 2^n t_3} + 1], \end{aligned} \tag{21}$$

$$\begin{aligned} B(2^{3^m 2^n t_2} + 1) = C(2^{3^m 2^n t_3} + 1)[2^{3^{m+1} 2^n t_3 (d_2 - 1)} - \\ -2^{3^{m+1} 2^n t_3 (d_2 - 2)} + \dots \\ \dots + 2^{3^{m+1} 2^n t_3 2} - 2^{3^{m+1} 2^n t_3} + 1]. \end{aligned} \tag{22}$$

Let $q > 1$ is prime number such, that

$$2^{3^m 2^n t_1} + 1 \equiv 0(mod\, q), \ C \equiv 0(mod\, q). \tag{23}$$

Then from (20) and (23) the comparisons follow

$$2^{3^m 2^n t_3 d_1} + 1 \equiv 0(mod\, q), \ 2^{3^{m+1} 2^n t_3} + 1 \equiv 0(mod\, q).$$

Let's consider number $b = 2^{3^m 2^n t_3}$. If $b \equiv 1(mod\, q)$, then $C = b^2 - b + 1 \equiv 1 \not\equiv 0(mod\, q)$. If $b \equiv -1(mod\, q)$, then $C \equiv 3 \equiv 0(mod\, q)$, $q = 3$, but as $n > 0$, that from (17) follows, that $C \not\equiv 0(mod\, 3)$. Have received the contradiction. Thus, $b^2 \not\equiv 1(mod\, q)$.

Let $l_0 > 1$ is an index, to which the number b^2 belongs on the module q, $(b^2)^{l_0} \equiv 1(mod\, q)$. As $(b^2)^{d_1} \equiv (b^2)^3 \equiv 1(mod\, q)$, that $d_1 \equiv 3 \equiv 0(mod\, l_0)$, $d_1 \equiv 0(mod\, 3)$. Have received the contradiction. Have proved, that $gcd(A, B) \equiv 0(mod\, C)$.

Let's assume, that there is an integer $d > 1$ such, that

$$A \equiv B \equiv 0(mod\, d),\ but\ gcd(C, d) = 1. \tag{24}$$

Then it follows from (20) and (24), that

$$2^{3^{m+1}2^{n+1}t_3d_1} - 1 \equiv 0(mod\, d),\ 2^{3^{m+1}2^{n+1}t_3d_2} - 1 \equiv 0(mod\, d).$$

Then $2^{3^{m+1}2^{n+1}t_3} - 1 \equiv 0(mod\, d)$, i.e.

$$(2^{3^{m+1}2^{n}t_3} + 1) \cdot (2^{3^{m+1}2^{n}t_3} - 1) \equiv 0(mod\, d). \tag{25}$$

If $gcd(2^{3^{m+1}2^{n}t_3} + 1, d) = d_0 > 1$, then it follows from (20), that $2^{3^{m}2^{n}t_3} \equiv -1(mod\, d_0)$, $A \equiv 3 \equiv 0(mod\, d_0)$, $d_0 = 3$. Have received the contradiction. Then from (25), (24) and (20) the comparisons follow $(2^{3^{m+1}2^{n}t_3} - 1) \equiv 0(mod\, d)$, $2^{3^{m+1}2^{n}t_3d_1} + 1 \equiv 0 \equiv 2(mod\, d)$. Have received the contradiction, since $d > 1$ is odd number. The equality (18) is proved.

2) Let's prove equality (19). Let's consider the following formulas

$$\begin{gathered} A_1 = 2^{2\cdot 3^{m_1}2^{n_1}t_1} - 2^{3^{m_1}2^{n_1}t_1} + 1, \\ A_1(2^{3^{m_1}2^{n_1}t_1} + 1) = 2^{3^{m_1+1}2^{n_1}t_3d_1} + 1, \\ A_2 = 2^{2\cdot 3^{m_2}2^{n_2}t_2} - 2^{3^{m_2}2^{n_2}t_2} + 1, \\ A_2(2^{3^{m_2}2^{n_2}t_2} + 1) = 2^{3^{m_2+1}2^{n_2}t_3d_2} + 1. \end{gathered} \tag{26}$$

Let's assume, that $q > 1$ is prime number such, that $A_1 \equiv A_2 \equiv 0(mod\, q)$. Then $2^{3^{m_1+1}2^{n_1}t_3d_1} + 1 \equiv 2^{3^{m_2+1}2^{n_2}t_3d_2} + 1 \equiv 0(mod\, q)$. Let $l_0 > 1$ is an index, to which the number 2 belongs on the module q. Then $3^{m_1+1}2^{n_1+1}t_3d_1 \equiv 3^{m_2+1}2^{n_2+1}t_3d_2 \equiv 0(mod\, l_0)$, i.e.

$$3^{m_1+1}2^{n_1+1}t_3 \equiv 3^{m_2+1}2^{n_2+1}t_3 \equiv 0(mod\, l_0). \tag{27}$$

2^a) Let's assume, that $m_1 < m_2$. Then from (27) we receive comparisons

$$3^{m_2}2^{n_2+1}t_3 \equiv 0(mod\, l_0),\ (2^{3^{m_2}2^{n_2}t_3} + 1) \cdot (2^{3^{m_2}2^{n_2}t_3} - 1) \equiv 0(mod\, q).$$

From the last comparison either $A_2 \equiv 3 \equiv 0(mod\, q)$, $q = 3$, or $A_2 \equiv 1 \not\equiv 0(mod\, q)$ follows. Have received the contradiction.

2^b) Let's assume, that $m_1 = m_2$, $n_1 < n_2$. Then from (27), (26) we receive comparisons

$$3^{m_2+1}2^{n_2}t_3 \equiv 0(mod\, l_0),\ (2^{3^{m_2+1}2^{n_2}t_3d_2} + 1) \equiv 0 \equiv 2(mod\, q).$$

Have received the contradiction. The theorem 4 is proved. $\square$

Corollary 2. *Let* $n \geq 1$ *is an integer,* $t > 1$ *is an odd number. Then the statements are fair :*

(1) If $t \not\equiv 0(mod\,3)$, *then*

$$L_3(2^n t) \equiv 0(mod\,L_3(2^n)).$$

Besides, $1 < L_3(2^n) < L_3(2^n t)$, *where* $L_3(2^n t)$ *is composite number.*

(2) If $t \equiv 0(mod\,3)$, *then*

$$gcd(L_3(2^n t), L_3(2^n)) = 1.$$

Corollary 3. *Let* $m \geq 0$, $n > 0$ *are integers;* $t > 1$ *is an odd number,* $t \not\equiv 0(mod\,3)$. *Then number* $L_3(3^m 2^n t)$ *is always composite number.*

The summary of the received results concerning composite numbers L_3.

Theorem 5 (About numbers L_3). *For numbers* L_3 *the statements are fair :*

(1) If $n \neq 3^m 2^n$, *where* $m \geq 0$, $n \geq 0$ *are integers, then number* $L_3(n)$ *is composite number.*

(2) Prime numbers $L_3(1) = L_3(2^0) = 3$, $L_3(2) = L_3(2^1) = 13$, $L_3(4) = L_3(2^2) = 241$, $L_3(32) = L_3(2^5) = 18446744069414584321$ *are known.*

Proof. □

For numbers $L_3(3^m 2^n)$, where $m \geq 0$, $n > 0$ are integers, the author has carried out the following check :

1) Numbers $L_3(2^n)$ at $6 \leq n \leq 15$ is composite;
2) Numbers $L_3(2 \cdot 3^m)$ at $1 \leq m \leq 8$ is composite;
3) Numbers $L_3(3 \cdot 2^n)$ at $1 \leq n \leq 12$ is composite;
4) Numbers $L_3(3^2 \cdot 2^n)$ at $1 \leq n \leq 11$ is composite;
5) Numbers $L_3(3^3 \cdot 2^n)$ at $1 \leq n \leq 9$ is composite.

5 Numbers $L_4(n) = 2^{2n} - 2^n - 1$

From **the statement 1** validity of comparisons follows

$$L_4(4k+3) \equiv 0(mod\,5),\ \ L_4(10k+2) \equiv L_4(10k+3) \equiv 0(mod\,11),$$

where $k \geq 0$ is any integer.

The numbers $L_4(n)$ are not free from squares, since

$$L_4(13) \equiv L_4(42) \equiv L_4(123) \equiv 0(mod\,11^2),$$
$$L_4(52) \equiv L_4(119) \equiv 0(mod\,19^2).$$

Prime numbers $L_4(n)$ and $L_4(n+1)$ are named **the prime L_4 number-twins**. **The author has found 4 pairs of the prime L_4 number-twins up to $n \leq 603$**, namely

$$L_4(1) = 1, L_4(2) = 11\ ;\ L_4(4) = 239, L_4(5) = 991\ ;$$
$$L_4(9) = 261631, L_4(10) = 1047551\ ;\ L_4(224), L_4(225).$$

6 Wonderful properties of *gcd* insularity

Definition 1 (Insularity to $\boldsymbol{gcd}$.). *Let $\boldsymbol{R_n \geq 0}$ is a sequence of integers, where $\boldsymbol{n > 0}$ is integer. $\boldsymbol{M}$ is a subset of natural numbers.*
Let's tell, that the sequence $\boldsymbol{R_n}$ on set $\boldsymbol{M}$ is isolated about $\boldsymbol{gcd}$, if the condition is fair :

$$\boldsymbol{gcd(R_n, R_m) = R_{gcd(n,m)}} \tag{28}$$

for $\forall \boldsymbol{n, m} \in \boldsymbol{M}$.

Corollary 4. *Let $\boldsymbol{k \geq 0}$ is integer, then numbers $\boldsymbol{L_1}$ on set*
$\boldsymbol{M_k = \{3^k \cdot t : t \geq 1}$ *is an odd number,* $\boldsymbol{t \not\equiv 0(mod\, 3)\}}$
are isolated about $\boldsymbol{gcd}$, i.e.

$$\boldsymbol{gcd(L_1(n), L_1(m)) = L_1(gcd(n, m))}$$

for $\forall \boldsymbol{n, m} \in \boldsymbol{M_k}$.

Corollary 5. *Let $\boldsymbol{k \geq 0, l > 0}$ are integers, then numbers $\boldsymbol{L_3}$ on set*
$\boldsymbol{M_{k,l} = \{3^k 2^l \cdot t : t \geq 1}$ *is an odd number,* $\boldsymbol{t \not\equiv 0(mod\, 3)\}}$
are isolated about $\boldsymbol{gcd}$, i.e.

$$\boldsymbol{gcd(L_3(n), L_3(m)) = L_3(gcd(n, m))}$$

for $\forall \boldsymbol{n, m} \in \boldsymbol{M_{k,l}}$.

Let's consider the generalized numbers repunit - integers of the following kind [7, 8, 14, 15, 16] :

$$\boldsymbol{M_n^{(b)} = (b^n - 1)/(b - 1),} \tag{29}$$

where $\boldsymbol{n \geq 1}$, $\boldsymbol{b \geq 2}$ are integers.

$$\boldsymbol{M_n^{+(b)} = (b^n + 1)/(b + 1),} \tag{30}$$

where $\boldsymbol{n \geq 1}$ is an odd number, $\boldsymbol{b \geq 2}$ is integer.

For the generalized numbers repunit (29) and (30) the theorem takes place.

Theorem 6. *Following formulas are fair :*

$$\boldsymbol{gcd(M_n^{(b)}, M_m^{(b)}) = M_{gcd(n,m)}^{(b)},} \tag{31}$$

where $\boldsymbol{n \geq 1}$, $\boldsymbol{b \geq 2}$ are integers.

$$\boldsymbol{gcd(M_n^{+(b)}, M_m^{+(b)}) = M_{gcd(n,m)}^{+(b)},} \tag{32}$$

where $\boldsymbol{n \geq 1}$ is an odd number, $\boldsymbol{b \geq 2}$ is integer.

Proof. 1) Let $(n,m)=d\geq 1$, where $n=n_1d$, $m=m_1d$, $(n_1,m_1)=1$. From definition (29) equalities follow

$$M_n^{(b)}=((b^d)^{n_1}-1)/(b-1)=M_d^{(b)}\cdot\{b^{d(n_1-1)}+\ldots+b^{2d}+b^d+1\},$$

$$M_m^{(b)}=((b^d)^{m_1}-1)/(b-1)=M_d^{(b)}\cdot\{b^{d(m_1-1)}+\ldots+b^{2d}+b^d+1\}.$$

Let

$$A=b^{d(n_1-1)}+\ldots+b^{2d}+b^d+1,\quad B=b^{d(m_1-1)}+\ldots+b^{2d}+b^d+1.$$

Lets assume, that $A\equiv\ B\equiv 0(mod\,q)$, where $q>1$ is prime number. Let $b_0=b^d$. If $b_0\equiv 1(mod\,q)$, then $n_1\equiv\ m_1\equiv 0(mod\,q)$. Have received the contradiction. Hence $b_0\not\equiv 1(mod\,q)$, then there exists an index $d_0>1$, to which the number b_0 belongs on the module q

$$(b^d)^{d_0}\equiv 1(mod\,q).$$

Then $n_1\equiv m_1\equiv 0(mod\,d_0)$. Have received the contradiction.

2) Let $(n,m)=d\geq 1$, where $n=n_1d$, $m=m_1d$ are odd numbers, $(n_1,m_1)=1$. From definition (30) equalities follow

$$M_n^{+(b)}=((b^d)^{n_1}+1)/(b+1)=$$

$$=M_d^{+(b)}\cdot\{b^{d(n_1-1)}-b^{d(n_1-2)}+\ldots+b^{2d}-b^d+1\},$$

$$M_m^{+(b)}=((b^d)^{m_1}+1)/(b+1)=$$

$$=M_d^{+(b)}\cdot\{b^{d(m_1-1)}-b^{d(m_1-2)}+\ldots+b^{2d}-b^d+1\}.$$

Let

$$A=b^{d(n_1-1)}-b^{d(n_1-2)}+\ldots+b^{2d}-b^d+1,$$

$$B=b^{d(m_1-1)}-b^{d(m_1-2)}+\ldots+b^{2d}-b^d+1.$$

Lets assume, that $A\equiv\ B\equiv 0(mod\,q)$, where $q>1$ is prime number. Let $b_0=b^{2d}$. If $b_0\equiv 1(mod\,q)$, then either $b^d\equiv 1(mod\,q)$, or $b^d\equiv -1(mod\,q)$. Then either $A\equiv 1\not\equiv 0(mod\,q)$, or $n_1\equiv\ m_1\equiv 0(mod\,q)$. Have received the contradiction.

Hence $b_0\not\equiv 1(mod\,q)$, then there exists an index $d_0>1$, to which the number b_0 belongs on the module q

$$(b^{2d})^{d_0}\equiv 1(mod\,q).$$

Since $(b^{2d})^{n_1}\equiv(b^{2d})^{m_1}\equiv 1(mod\,q)$, then $n_1\equiv m_1\equiv 0(mod\,d_0)$. Have received the contradiction. □

Corollary 6. *Let $\mathbb{P}$ is a set of all positive integers, $\mathbb{O}$ is a set of all odd numbers.*
*(1) Numbers $M_n^{(b)}$ on set $\mathbb{P}$ are isolated about **gcd**, i.e.*

$$gcd(M_n^{(b)}, M_m^{(b)}) = M_{gcd(n,m)}^{(b)}$$

for $\forall n, m \in \mathbb{P}$.
*(2) Numbers $M_n^{+(b)}$ on set $\mathbb{O}$ are isolated about **gcd**, i.e.*

$$gcd(M_n^{+(b)}, M_m^{+(b)}) = M_{gcd(n,m)}^{+(b)}$$

for $\forall n, m \in \mathbb{O}$.

7 The open problems of numbers $L(n)$

Author offers some open problems, as the unsolved tasks concerning numbers $L(n) = 2^{2n} \pm 2^n \pm 1$.
Problem 1. *Whether there are prime numbers $L_1(3^k)$ for $k > 2$?*

The author has checked up, that the numbers $L_1(3^k)$ for $k = 3, 4, 5, 6, 7, 8, 9, 10$ are composite !

Problem 2. *Whether there are infinitely many prime numbers $L_2(p)$, where p is prime number ?*

Problem 3. *Whether there are prime numbers $L_2(2^n)$ for $n > 17$?*

Problem 4. *Whether there are prime numbers $L_3(2^n)$ for $n > 5$?*

Problem 5. *Whether there are prime numbers $L_3(3^m \cdot 2^n)$ for $m \geq 1$, $n > 1$?*

Problem 6. *Whether there are infinitely many prime numbers-twins $L_4(n)$, $L_4(n+1)$, where $n \geq 1$?*

8 Conclusion "The concrete theory of numbers"

It is necessary to explain the title of article "The concrete theory of numbers". Having had a look in the Wladimir Igorewitsch Arnold foreword "From Fibonacci up to Erdös" to the remarkable book of Ronald Graham, Donald Knuth and Oren Patashnik "The Concrete mathematics"[5], it is possible to answer the question : **what is "the concrete theory of numbers" ?**

The theories come and leave, but natural series of numbers remains and constantly generates new complicated problems. The new theories are again created for their decision. The process cannot be stopped **:)**

The concrete theory of numbers created by titanic efforts of Pierre de Fermat and Leonhard Euler is an art to solve riddles of a natural series of numbers. It is

enough to look to the tasks list [11], which has been put and decided by Pierre de Fermat, or to get acquainted with tasks, which has been decided, investigated and propagandized by Waclaw Sierpinski [3],to be convinced **- a natural series of numbers doesn't drowse, it is always ready to a human challenge!**

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References

[1] **Arnold I.V.** *Teoriya chisel. - M. : Uchpedgiz, 1939.*

[2] **Vinogradov I. M.** *Osnovy teorii chisel. - M. : Nauka, 1981.*

[3] **Serpinskii V.** *250 zadach po elementarnoi teorii chisel. - M. : Prosveshenie, 1968.*

[4] **Ronald L. Graham, Donald E. Knuth, Oren Patashnik.** *Concrete Mathematics : A Foundation for Computer Science, 2nd edition (Reading, Massachusetts: Addison-Wesley), 1994.*

[5] **Grekhem R., Knut D., Patashnik O.** *Koncretnay matematika. Osnovanie informatiki : Per. s angl. - M. : Mir, 1998.*

[6] **Weisstein, Eric W.** *"Fermat Number". From MathWorld–A Wolfram Web Resource. —http://mathworld.wolfram.com/FermatNumber.html/. ©1999—2007 Wolfram Research, Inc.*

[7] **Weisstein, Eric W.** *"Mersenne Number". From MathWorld–A Wolfram Web Resource. —http://mathworld.wolfram.com/MersenneNumber.html/. ©1999—2007 Wolfram Research, Inc.*

[8] **Weisstein, Eric W.** *Repunit. From MathWorld–A Wolfram Web Resource. —http://mathworld.wolfram.com/Repunit.html/. ©1999—2008 Wolfram Research, Inc.*

[9] **Mersenne's conjecture – Wikipedia, the free encyclopedia –** *http://en.wikipedia.org/wiki/Mersenne conjectures/.*

[10] **Sloane, N. J. A.** *Sequence A000225 in "The On-Line Encyclopedia of Integer Sequences."*

[11] ***Zadachi Ferma.*** *—http://neves.suncloud.ru/task/fermat.htm*

[12] ***GP/PARI CALCULATOR*** *is free software Version 2.3.0. —http://pari.math.u-bordeaux.fr/*

[13] **Bateman, P. T., Selfridge, J. L., Wagstaff, Jr. Samuel S.** *"The new Mersenne conjecture". American Mathematical Monthly 96: 125-128.*

[14] **Dubner, H.** *"Generalized Repunit Primes." Math. Comput. 61, 927-930, 1993.*

[15] **Dubner, H. and Granlund, T.** *"Primes of the Form* $(b^n + 1)/(b + 1)$ *." J. Int. Sequences 3, No. 00.2.7, 2000. http://www.cs.uwaterloo.ca/journals/JIS/VOL3/DUBNER/dubner.html.*

[16] **Granlund, T.** *"Repunits." http://www.swox.com/gmp/repunit.html.*

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